EXERCISES [MAI 1.8]

COMPLEX NUMBERS (CARTESIAN FORM)

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (a) $\Delta = -16$

(b) $z = \dfrac{8 \pm 4i}{2} = 4 \pm 2i$

(c) $(z - 4 - 2i)(z - 4 - 2i) = (z-4)^2 - 4i^2 = (z-4)^2 + 4$

2. (a) $\Delta = -144$

$z = \dfrac{8 \pm 12i}{8} = \dfrac{8}{8} \pm \dfrac{12}{8}i = 1 \pm \dfrac{3}{2}i$

(b) (i) $S = 1 + \dfrac{3}{2}i + 1 - \dfrac{3}{2}i = 2$ and $-\dfrac{b}{a} = \dfrac{8}{4} = 2$

(ii) $S = \left(1 + \dfrac{3}{2}i\right) + \left(1 - \dfrac{3}{2}i\right) = 1^2 - \left(\dfrac{3}{2}i\right)^2 = 1 + \dfrac{9}{4} = \dfrac{13}{4}$ and $\dfrac{c}{a} = \dfrac{13}{4}$

3. (a) $(1 - i\sqrt{3})^2 = 1^2 - 2\sqrt{3}i + \left(\sqrt{3}i\right)^2 = -2 - 2\sqrt{3}i$

(b) $(1 - i\sqrt{3})^3 = (1 - i\sqrt{3})^2(1 - i\sqrt{3}) = \left(-2 - 2i\sqrt{3}\right)(1 - i\sqrt{3}) = -2 - 6 - 2i\sqrt{3} + 2i\sqrt{3} = -8$

4. $z = 1 + \dfrac{i(i + \sqrt{3})}{(i - \sqrt{3})(i + \sqrt{3})} = 1 + \dfrac{i(i + \sqrt{3})}{-4} = \dfrac{-5 + i\sqrt{3}}{-4} = \dfrac{5}{4} - \dfrac{i\sqrt{3}}{4}$

5. $z = \dfrac{2}{(1 - i)} \cdot \dfrac{(1 + i)}{(1 + i)} + 1 - 4i = 1 + i + 1 - 4i = 2 - 3i$

$z^2 = (2 - 3i)^2 = 4 - 12i - 9 = -5 - 12i$

6. (a) $a - 2 = 7 \Leftrightarrow a = 9$

$b - 1 = 3 \Leftrightarrow b = 4$

(b) $c - 2 = 0 \Leftrightarrow c = 2$

$d - 1 = 0 \Leftrightarrow d = 1$

7. The final result is 3+i

8. $(a + i)(2 - bi) = 7 - i \Rightarrow 2a - abi + 2i - bi^2 = 7 - i$

$\Rightarrow 2a - abi + 2i + b = 7 - i$

$2a + b = 7 \Leftrightarrow b = 7 - 2a$

$2 - ab = -1 \Leftrightarrow ab = 3$

Hence

$a(7 - 2a) = 3 \Leftrightarrow 2a^2 - 7a + 3 = 0 \Rightarrow a = 3$ and $b = 1$

9. $2(p + iq) = q - ip - 2(1 - i)$

$2p = q - 2$

$2q = -p + 2$

$p = -0.4$, $q = 1.2$

10. Let $z = x + \mathrm{i}y$

$(1 - \mathrm{i})(x + \mathrm{i}y) = 1 - 3\mathrm{i}$

$x + y - \mathrm{i}(x - y) = 1 - 3\mathrm{i}$

$$\left.\begin{array}{l} x + y = 1 \\ x - y = 3 \end{array}\right\} \Rightarrow x = 2,\ y = -1$$

OR

$(1 - \mathrm{i})z = 1 - 3\mathrm{i} \Leftrightarrow z = \dfrac{1-3\mathrm{i}}{1-\mathrm{i}} \Leftrightarrow z = \dfrac{1-3\mathrm{i}}{1-\mathrm{i}} \times \dfrac{1+\mathrm{i}}{1+\mathrm{i}} \Leftrightarrow z = 2 - \mathrm{i}$

$\Rightarrow x = 2,\ y = -1$

11. $(a + b\mathrm{i})(2 - \mathrm{i}) = (5 - \mathrm{i})$

$(a + b\mathrm{i}) = \dfrac{(5-\mathrm{i})}{(2-\mathrm{i})} = \dfrac{11}{5} + \dfrac{3}{5}\mathrm{i}$ (using a graphic display calculator)

Therefore $a = \dfrac{11}{5}$, $b = \dfrac{3}{5}$

OR $(a + b\mathrm{i})(2 - \mathrm{i}) = (5 - \mathrm{i})$

$(2a + b) + (2b - a)\mathrm{i} = (5 - \mathrm{i})$

$2a + b = 5 \qquad -a + 2b = -1$

therefore $a = \dfrac{11}{5}$, $b = \dfrac{3}{5}$

12.

$$\frac{z(3-4\mathrm{i})}{(3+4\mathrm{i})(3-4\mathrm{i})} - \frac{5\mathrm{i}(z-1)}{-5\times 5\mathrm{i}\times\mathrm{i}} = \frac{5(3+4\mathrm{i})}{(3-4\mathrm{i})(3+4\mathrm{i})}$$

$$z(3-4\mathrm{i}) - 5\mathrm{i}(z-1) = 15 + 20\mathrm{i}$$

$$z(3-4\mathrm{i}-5\mathrm{i}) = 15 + 20\mathrm{i} - 5\mathrm{i}$$

$$z = \frac{15+15\mathrm{i}}{3-9\mathrm{i}}$$

$$z = -1 + 2\mathrm{i}$$

13.

METHOD 1

$$\frac{a}{1+\mathrm{i}} + \frac{b}{1-2\mathrm{i}} = a\left(\frac{1}{2} - \frac{1}{2}\mathrm{i}\right) + b\left(\frac{1}{5} + \frac{2}{5}\mathrm{i}\right)$$

$\dfrac{a}{2} + \dfrac{b}{5} = 3$ and $-\dfrac{a}{2} + \dfrac{2b}{5} = 0$

Solving gives $a = 4,\ b = 5$.

METHOD 2

$$\frac{a}{1+\mathrm{i}} + \frac{b}{1-2\mathrm{i}} = 3$$

$$a(1-2\mathrm{i}) + b(1+\mathrm{i}) = 3(1-2\mathrm{i})(1+\mathrm{i})$$

$$= 9 - 3\mathrm{i}$$

$\mathrm{Re}(z): a + b = 9$

$\mathrm{Im}(z): -2a + b = -3$

$3a = 12$

$a = 4,\ b = 5$

14. $\mathrm{i}(z + 2) = 1 - 2z \Rightarrow (2 + \mathrm{i})z = 1 - 2\mathrm{i}$

$\Rightarrow z = \dfrac{1-2\mathrm{i}}{2+\mathrm{i}} = \dfrac{1-2\mathrm{i}}{2+\mathrm{i}} \times \dfrac{2-\mathrm{i}}{2-\mathrm{i}} = \dfrac{-5\mathrm{i}}{5} = -\mathrm{i}.$ $\quad (a = 0,\ b = -1)$

15.

Solving simultaneously

$2z_1 + 3z_2 = 7$

$2z_1 + 2iz_2 = 8 + 8i$

$z_2(2i - 3) = 1 + 8i$

$z_2 = \dfrac{1+8i}{2i-3} = 1 - 2i$

$z_1 = 4 + 4i - i(1 - 2i) = 2 + 3i$

16. Let $z = a + bi$, so $z^* = a - bi$ $\quad |z|^2 = a^2 + b^2 = 20$

$$\frac{25}{a+bi} - \frac{15}{a-bi} = 1 - 8i \Rightarrow \frac{25(a-bi) - 15(a+bi)}{a^2+b^2} = 1 - 8i$$

$$\frac{10a}{20} = 1 \Rightarrow a = 2 \qquad -\frac{40b}{20} = -8 \Rightarrow b = 4$$

$z = 2 + 4i$

17.

METHOD 1

Substituting $z = x + iy$ to obtain $w = \dfrac{x+yi}{(x+yi)^2 + 1}$

$$w = \frac{x+yi}{x^2 - y^2 + 1 + 2xyi}$$

Use of $(x^2 - y^2 + 1 - 2xyi)$ to make the denominator real.

$$= \frac{(x+yi)(x^2 - y^2 + 1 - 2xyi)}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

$$\text{Im}\, w = \frac{y(x^2 - y^2 + 1) - 2x^2y}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

$$= \frac{y(1 - x^2 - y^2)}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

$\text{Im}\, w = 0 \quad \Rightarrow 1 - x^2 - y^2 = 0$ *i.e.* $|z| = 1$ as $y \neq 0$

METHOD 2

$w(z^2 + 1) = z$

$w(x^2 - y^2 + 1 + 2ixy) = x + yi$

Equating real and imaginary parts

$w(x^2 - y^2 + 1) = x$ and $2wx = 1,\ y \neq 0$

Substituting $w = \dfrac{1}{2x}$ to give $\dfrac{x}{2} - \dfrac{y^2}{2x} + \dfrac{1}{2x} = x$

$-\dfrac{1}{2x}(y^2 - 1) = \dfrac{x}{2}$ or equivalent

$x^2 + y^2 = 1$, *i.e.* $|z| = 1$ as $y \neq 0$

18. (a) $(1 + i)^2 = 1 + 2i + i^2 = 2i$

(b) $(a + ai)^2 = a^2(1 + i)^2 = 2a^2 i$

(c) $(a + ai)^4 = (2a^2 i)^2 = -4a^4$

(d) $(a + ai)^4 = (-4a^4)^2 = 16a^8$

B. Paper 2 questions (LONG)

19. The final results are

(a) $(2+5i)z=-6+14i \Leftrightarrow z=\dfrac{-6+14i}{2+5i}=2+2i$ 2+2i

(b) $(2+5i)z+9=3z+19i \Leftrightarrow (2+5i-3)z=-9+19i$

$\Leftrightarrow (-1+5i)z=-9+19i \Leftrightarrow z=\dfrac{-9+19i}{-1+5i}$

$\Leftrightarrow z=4+i$

(c) Let $z=x+yi$ and $\bar{z}=x-yi$

Then

$(2+5i)(x+yi)+8=3(x-yi)+20i$

$2x+2yi+5xi-5y+8=3x-3yi+20i$

Equal real parts: $2x-5y+8=3x \Leftrightarrow x+5y=8$

Equal imaginary parts: $2y+5x=-3y+20 \Leftrightarrow 5x+5y=20 \Leftrightarrow x+y=4$

$x=3,\quad y=1$

Hence $z=3+i$

20. theoretical

21. (a) $w = x + 16 + iy$

(i) the conjugate is $x + 16 - iy$

(ii) $|w|^2=(x+16)^2+y^2=x^2+y^2+32x+256$

(b) $|z + 1|^2 = (x + 1)^2 + y^2 = x^2+y^2+2x+1$

(c) $|z+16|=4|z+1| \Leftrightarrow |z+16|^2=16|z+1|^2$

$\Rightarrow x^2 + y^2 + 32x + 256 = 16x^2 + 16y^2 + 32x + 16$

$\Rightarrow 15x^2 + 15y^2 = 240$

$\Rightarrow x^2 + y^2 = 16$

Therefore, $|z| = \sqrt{x^2+y^2} = 4$.